Wave field determination using tomography of the ambiguity function

Jinhong Tu* and Shinichi Tamura*
Department of Informatics and Mathematical Science, Osaka University, Osaka, Japan
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Ambiguity function (AF) theory is proposed to reconstruct a complex wave field using tomography by measurements of intensity in a refractive optical system. By performing one-dimensional (1D) inverse Fourier transforms of intensities with some adjustment in various longitudinal optical system parameters, the corresponding AF values along the lines at different angles in the AF phase space are obtained; therefore, the mutual intensity function is reconstructed by performing a 1D Fourier transform of the reconstructed AF values. The reconstruction process in some cases is considered to be simpler than the equivalent theory using Wigner distribution function. [S1063-651X(97)08802-8]

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The determination of a complex wave field with a partially coherent or fully coherent state by measurements of intensity only is currently a subject of considerable interest. The wave field may represent either a scalar electromagnetic field [1–5] or the quantum mechanical wave function of a matter wave [6–10], since there exists a mathematically analogous description between them [3]. For a scalar electromagnetic quasimonochromatic field, the mutual intensity $\Gamma(\mathbf{r},\mathbf{r}')$, the second-order statistics of the wave field denoted by the complex amplitude $\Psi(\mathbf{r})$, can be written as

$$\Gamma(\mathbf{r},\mathbf{r}')=\langle\Psi(\mathbf{r})\Psi^*(\mathbf{r}')\rangle, \tag{1}$$

where the brackets indicate an ensemble average over the set of realizations of the function $\Psi(\mathbf{r})$. In the case of a quantum mechanical wave function $\Psi(\mathbf{r})$, the second-order statistics of the wave function $\Gamma(\mathbf{r},\mathbf{r}')$, also defined by Eq. (1), is the density matrix. The second-order statistics function $\Gamma(\mathbf{r},\mathbf{r}')$ provides important characterization of the wave field in both cases. It is well known that an equivalent representation of this second-order statistics function in both cases is the Wigner distribution function (WDF) which is defined in phase space [1,3–6,11].

Since Smithey *et al.* [6] first succeeded in determining the density matrix from the reconstructed WDF using phase-space tomography, much progress in a quantum mechanical context has been made [10]. At the same time, a method using phase-space tomography from intensity measurements in a refractive optical system was presented, owing to the analogy between quantum and classical waves, for reconstructing the full WDF [3,4]. The method is based on a rotation of the WDF in phase space and determination of a projected function for a sufficient set of values of the rotation angle. By means of fractional-order Fourier transform theory [3] or the generalized Fresnel transform [4] theory the projected WDF of different rotation angles is related to the intensity for the adjustment of different longitudinal optical system parameters. As a result, the WDF is normally reconstructed by using the filtered back-projection algorithm for the inverse Radon transform [12]. The method can be applied in the cases of fully or partially coherent resources. An advantage of the method is that the data analysis requires no deconvolution and is noniterative. Another advantage is that the refractive optics method does not use diffractive element so that the field power is less lost and the reconstructed wave field maintains rather higher-order spatial frequency. The method is experimentally applied to determination of the spatially varying amplitude and phase of a quasimonochromatic optical field [5] and can also find its application in atom optics [9].

It is also known that the WDF is not confined to be a positive value and it cannot be interpreted as intensity or probability distribution, therefore the WDF itself cannot be measured directly, only its marginal (projected) functions are experimentally accessible. Projection of the WDF at arbitrary angles yields the Radon transform of the WDF. Hence the WDF can be obtained by the inverse Radon transform of its marginal (projected) function. Although the filtered back-projection algorithm for the inverse Radon transform provides high-quality reconstructed data [12,13] and is the most popular, a considerable amount of computation time is required primarily due to the back-projection operation. Furthermore, usually an additional one-dimensional (1D) Fourier transform must be performed on the reconstructed 2D WDF to obtain mutual intensity function (or the density matrix), especially in the application to optical phase retrieval [5]. Therefore it is necessary to develop a new algorithm where the detour via the WDF is avoided.

In this paper we propose a theory of another well-known phase-space function, namely, ambiguity function (AF) [14–16], to reconstruct an optical complex wave field by measurements of intensity using the first-order optical system as suggested in Refs. [3,4]. As is known, AF provides a simple approach for treatment of a partially coherent system [16], which was from the uncertain principle of distance and speed of the object in the radar measurement and had been generalized to be employed in the Fourier optics. AF also cannot be measured directly, but the AF at a special value is related to the intensity through an inverse Fourier transformation. With the full knowledge of the AF in the AF phase space using tomography, the mutual intensity can be recovered according to the definition of AF.

*Also at Division of Functional Diagnostic Imaging, Medical School, Osaka University, Osaka, Japan.

For simplicity we only take the 1D case into consideration. We believe that the 2D case can easily be extended in the same manner as proposed in Refs. [3,4]. Also we omit some unimportant constants before the integrals in the following deduction which do not affect the result. Therefore 1D mutual intensity can be expressed in terms of the center and difference coordinates x and Δx, respectively, $J(x,\Delta x)$ as

$$J(x,\Delta x)=\Gamma(x_1,x_2), \tag{2}$$

where

$$x=(x_1+x_2)/2, \tag{3}$$

and

$$\Delta x=x_1-x_2. \tag{4}$$

By definition AF is expressed in the form of Fourier transformation of a mutual intensity $J(x,\Delta x)$ by the notation

$$A(\Delta\nu,\Delta x)=\int J(x,\Delta x)\exp(-i2\pi\Delta\nu x)dx, \tag{5}$$

where $\Delta\nu$ is the difference of spatial frequency ν. In the above and following deductions the integral is from $-\infty$ to ∞ without declaration.

Since the mutual intensity $J(x,\Delta x)$ can be expressed by a WDF through a Fourier transformation as

$$J(x,\Delta x)=\int W(x,\nu)\exp(i2\pi\Delta x\nu)d\nu x. \tag{6}$$

By substituting Eq. (6) into Eq. (5), the relation between the AF and the WDF can be expressed by

$$A(\Delta\nu,\Delta x)=\int\int W(x,\nu)\exp[i2\pi(\Delta x\nu-x\Delta\nu)]dxd\nu. \tag{7}$$

Like WDF, in the Fresnel approximation, the propagation of AF through a distance or a transparent inhomogeneous medium, such as a thin lens or a quadratic graded index (GRIN) fiber, can be described by a certain transport matrix $M(z)$ as the so-called $\mathcal{ABCD}$ matrix,

$$M(z)=\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{bmatrix}. \tag{8}$$

The WDF and the AF at distance z are related to the WDF and the AF at $z=0$ by

$$W(x,\nu,z)=W(\mathcal{A}x+\mathcal{B}\nu,\mathcal{C}x+\mathcal{D}\nu,0), \tag{9}$$

and

$$A(\Delta\nu,\Delta x,z)=A(\mathcal{D}\Delta\nu+\mathcal{C}\Delta x,\mathcal{B}\Delta\nu+\mathcal{A}\Delta x,0). \tag{10}$$

In other words, in a phase space such as WDF or AF the propagation through the medium that can be described by $\mathcal{ABCD}$ matrix $M(z)$ does not change the values of WDF or AF, only the coordinates (x,ν) or $(\Delta\nu,\Delta x)$ are modified by affine transformations as expressed in relation (11) or (12),

$$\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{bmatrix}\begin{bmatrix}x\\ \nu\end{bmatrix}=\begin{bmatrix}x'\\ \nu'\end{bmatrix}, \tag{11}$$

$$\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta\nu\end{bmatrix}=\begin{bmatrix}\Delta x'\\ \Delta\nu'\end{bmatrix}. \tag{12}$$

Inversely, we rewrite Eq. (10) through replacing the parameter $\Delta\nu,\Delta x$ by $\Delta\nu',\Delta x'$, respectively, according to Eq. (12), interchanging the position of two sides of Eq. (10), and then dropping the prime from parameters $\Delta\nu'$ and $\Delta x'$ for the sake of convenience, therefore the AF at distance $z=0$ is related to the AF at z by

$$A(\Delta\nu,\Delta x,0)=A(d\Delta\nu+c\Delta x,b\Delta\nu+a\Delta x,z), \tag{13}$$

where a, b, c, and d are the components of the inverse matrix of the $\mathcal{ABCD}$ matrix. Since the $\mathcal{ABCD}$ matrix has the following property as

$$\mathcal{AD}-\mathcal{BC}=1, \tag{14}$$

it is easy to show that

$$ad-bc=1 \tag{15}$$

and

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}\mathcal{D} & -\mathcal{B}\\ -\mathcal{C} & \mathcal{A}\end{bmatrix}. \tag{16}$$

As the mutual intensity at z can be transformed into intensity at z as

$$I(x,z)=J(x,\Delta x=0,z), \tag{17}$$

the light intensity distribution function can be obtained through a Fourier transform of AF $A(\Delta\nu,\Delta x=0,z)$ at z by

$$I(x,z)=\int A(\Delta\nu,\Delta x=0,z)\exp(i2\pi\Delta\nu x)d\Delta\nu. \tag{18}$$

Conversely, $A(\Delta\nu,0,z)$ can also be obtained through an inverse Fourier transform of $I(x,z)$ as

$$A(\Delta\nu,0,z)=\int I(x,z)\exp(-i2\pi\Delta\nu x)x. \tag{19}$$

If we let

$$\Delta x=-b\Delta\nu/a, \tag{20}$$

then put it in Eq. (13) and use Eq. (15) to obtain

$$A(\Delta\nu,-b\Delta\nu/a,0)=A(\Delta\nu/a,0,z), \tag{21}$$

we can establish a relation between the intensity at z and the AF at $z=0$ from Eq. (19) with the condition of Eq. (20), which yields

$$A(\Delta\nu,-b\Delta\nu/a,0)=\int I(x,z)\exp(-i2\pi\Delta\nu x/a)dx, \tag{22}$$

where $I(x,z)$ can also be regarded as a function of parameters a and b.

According to Eq. (2), mutual intensity $J(x,\Delta x)$ at $z=0$ can be seen as a Fourier transform of $A(\Delta\nu,\Delta x,0)$ as

$$J(x,\Delta x)=\int A(\Delta\nu,\Delta x,0)\exp(i2\pi\Delta\nu x)d\Delta\nu. \quad (23)$$

Therefore Eqs. (20)–(23) yield another simple way of unique determination of a complex wave field using tomography through measurements of intensity only. Let us consider the phase-space $(\Delta\nu,\Delta x)$ plane, and let $-b/a$ have any fixed value. From supposed knowledge of $I(x,z)$ we pass to $A(\Delta\nu,\Delta x=0,z)$ by an inverse Fourier transformation [see Eq. (19)]. This in turn yields the values of $A(\Delta\nu,\Delta x,z=0)$ along the line given by Eq. (20) through the origin and at an angle with the $\Delta\nu$ axis θ, i.e.,

$$\tan\theta=-b/a=\mathcal{B}/\mathcal{D}. \quad (24)$$

By varying $\mathcal{B}$ and/or $\mathcal{D}$ to let θ be $[0,\pi)$ such a line sweeps the $(\Delta\nu,\Delta x)$ plane by rotating around the origin, therefore fills up phase space of AF. Finally, by Fourier transformation of $A(\Delta\nu,\Delta x,z=0)$ [see Eq. (23)] the mutual intensity can be determined. This method can be regarded as a direct Fourier reconstruction technique. As is pointed out by Refs. [12,13], the direct Fourier reconstruction technique requires a fast and accurate polar to Cartesian coordinate interpolation if equiangle sampling is adopted. The quality of the reconstructed data is often dependent on the accuracy of the interpolation. Since the density of the radial points becomes sparser as one gets farther away from the center, the interpolation error becomes larger. This will result in the error in the calculation of the high-frequency components. However, unlike the x-ray computer tomography (CT), the method proposed in this paper is not necessary to require equiangle sampling; the sampling interval can be controlled by adjustment of optical longitude system parameters. Data acquisition schemes that lead to 1D interpolation are possible [13]. Similar to NMR reconstruction, if equisampling in the Cartesian domain is applied, the calculation is simplified at the cost of a large amount of data acquisition and the higher spatial frequency components will be retained.

Now let us consider two special systems which can be described by the $\mathcal{ABCD}$ matrix. First, in a free space propagation case, the $\mathcal{ABCD}$ matrix is given by

$$M(z)=\begin{bmatrix}1 & -\lambda z\\ 0 & 1\end{bmatrix}, \quad (25)$$

where λ is the wavelength of the light. To let θ be $[0,\pi)$ requires z to span the whole axis. Thus the method using this system to recover a complex wave field seems to be impractical. The result dealing with a free space propagation has been discussed in Refs. [1,2].

Second, in the case of a refractive lens system [3,4], the $\mathcal{ABCD}$ matrix (the component may be different with respect to different definitions in the literature) is given by

$$M(z)=\begin{bmatrix}1-Z_2/f & \lambda Z_2(1-Z_1/f)+\lambda Z_1\\ 1/\lambda f & 1-Z_1/f\end{bmatrix}, \quad (26)$$

where f is the focal length of the lens, Z_1 and Z_2 are the distances to the lens from the input plane and output plane, therefore we obtain the scaling factor

$$\{(1-Z_1/f)^2+[\lambda Z_2(1-Z_1/f)+\lambda Z_1]^2\}^{1/2} \quad (27)$$

and the angle

$$\theta=\arctan[\lambda Z_2+\lambda Z_1/(1-Z_1/f)], \quad (28)$$

which is proved below to be equal to the rotation angle of the WDF phase space. By suitable adjustment of the parameters f, Z_1, and Z_2, the AF values in the whole space can be obtained. It is worth mentioning that the equisampling in the AF Cartesian domain can be realized if f is easily controlled at a very small interval and Z_1 and Z_2 are managed to be several fixed values.

To show the validity of the theory proposed in this paper we prove that the method is equivalent to the WDF method proposed in Refs. [3,4] when a first-order optical system such as free space, lens, GRIN media, or combinations of them is adopted. It is worth noting that the relation between fractional Fourier transform and AF is discussed in Ref. [17]. Now we assume that an angle α defined in WDF space has the same value as θ defined in AF space [see Eq. (24)], then

$$\cos\alpha=a/\sqrt{a^2+b^2}, \quad (29)$$

and letting

$$x=X\sqrt{a^2+b^2}, \quad (30)$$

the right side of Eq. (22) can be written as

$$\int I(x,z)\exp(-i2\pi\Delta\nu x/a)dx$$
$$=\sqrt{a^2+b^2}\int I(X,z)\exp(-i2\pi\Delta\nu X/\cos\alpha)dX. \quad (31)$$

At the same time, using the relation between AF and WDF described by Eq. (7), the left side of Eq. (22) can be deduced as

$$A(\Delta\nu,-b\Delta\nu/a,0)=\int\int W(X,K,0)$$
$$\times\exp(-i2\pi\Delta\nu X/\cos\alpha)dKdX, \quad (32)$$

where

$$\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}x\\ \nu\end{bmatrix}=\begin{bmatrix}X\\ K\end{bmatrix}. \quad (33)$$

From Eqs. (31) and (32) therefore, we obtain a relation in the form of Radon transformation as

$$\sqrt{a^2+b^2}I(X,z)=\int W(X,K,0)dK. \quad (34)$$

Equations (9), (33), and (34) are supposed to be the basic relations of the WDF method [3,4]. The relations show that the light intensity $I(X,z)$ at plane $z=$const can be considered as the projection of the function $W(X,K,0)$ through particular orientations in the (X,K) plane, given from the linear transform [see Eq. (9)]. Those projections are scaled by factor $\sqrt{a^2+b^2}$ determined from the transfer matrix of the medium. By moving to a different z plane $W(X,K,0)$ becomes rotated by a certain angle α with an equal value to θ in AF space and so the measured intensity corresponds to a suitably rotated projection (apart from scaling). Hence the WDF is reconstructed by using the filtered back-projection algorithm for the inverse Radon transform with a sufficient number of projections.

In conclusion, in this paper we propose the AF theory to reconstruct a complex wave field using tomography by measurements of intensity in a refractive optical system. We also prove that the AF theory is equivalent to the WDF theory proposed in Refs. [3,4]. The theory using WDF requires the filtered back-projection algorithm for the inverse Radon transform while the AF theory only resorts to Fourier transformation and may be accomplished by a fast digital algorithm. It is found that the theory proposed in this paper is something like the Fourier reconstruction algorithm adopted in x-ray computer tomography and in nuclear magnetic resonance [13]. In most cases, the representation of the phase-space function, such as WDF or AF, only plays an intermediate role, and the mutual intensity is often required, hence, unlike the normal Fourier reconstruction algorithm in projection reconstruction applied to a system such as x-ray CT in which the steps of a 1D Fourier transformation and a 2D inverse Fourier transformation are adopted to reconstruct image, the AF theory allows the reconstruction of the mutual intensity function only through the steps of a 1D inverse Fourier transformation of intensities and then a 1D Fourier transformation of the reconstructed AF. It may also be found that the AF plays an analogous role to the expectation value of the displacement operator in a quantum mechanical context for determining the density matrix [7]. As is known, the relations among the rotation angle and system parameters are not linear, therefore in the experiment, the WDF method using a filtered back-projection algorithm with equiangle sampling for the inverse Radon transform seems to encounter the same difficulty as the AF method without equiangle sampling. Nevertheless, in the application of atom optics, the variation of system parameters can be made by varying focal length (by modulating the laser power, for example) [9] hence the measured data can probably be manipulated [13] so as to realize 1D interpolation or even by equisampling in Cartesian domain without additional interpolation. Thus the reconstruction process can be simplified. The theory proposed in this paper also provides another perspective view to complementarily understand the physics of complex wave reconstruction by intensity measurements.

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